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Eigenvalue bounds in linear inviscid stability theory

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New eigenvalue bounds are derived for the linear stability of inviscid parallel flows, both for homogeneous and for stratified fluids. The usefulness of these bounds, as compared with that of previous results, is assessed for several examples. For homogeneous fluids the new upper bounds for the imaginary part c_i of the complex phase velocity are sometimes better than previous criteria. For both homogeneous and stratified flows, the new upper bounds for the wavenumber α of neutrally stable disturbances improve on previous results, giving values within 10 % of the known exact solution in several cases.

1. Introduction

Rayleigh's equation governing the linear stability of inviscid parallel flows is

$$(\overline{u}-c)(\phi''-\alpha^2\phi)-\overline{u}''\phi = 0, \tag{1.1}$$

where $\overline{u}(z)$ is the primary velocity profile, primes denote d/dz, α is the wavenumber and $c \equiv c_r+ic_i$ is the complex phase velocity of a small wavelike disturbance with stream function of the form $\psi = \phi(z)\, e^{i\alpha(x-ct)}$. If the flow is confined by plane boundaries situated at $z = a$ and $z = b$ the appropriate boundary conditions are $\phi(a) = \phi(b) = 0$. These boundary conditions and equation (1.1) define a complex eigenvalue problem for c as a function of α (or vice versa), given the velocity distribution $\overline{u}(z)$. In particular, if $c_i > 0$ for some positive α, the flow is unstable while, if $c_i \leqslant 0$ for *all* positive α, the flow is stable. In fact, the Rayleigh equation is a valid approximation only for $c_i \geqslant 0$, so any results deduced from this equation will apply only to unstable or neutrally stable disturbances (see Lin 1955, § 8).

It is instructive to seek general results which yield bounds for c, particularly since $\max_\alpha (\alpha c_i)$ is the largest temporal growth rate of a wavelike disturbance. A number of such results have been derived previously (see Drazin & Howard 1966), the most important being the following:

(i) If $c_i > 0$, $\overline{u}''$ must change sign somewhere in $[a,b]$.

(ii) Fjørtoft's criterion. If $c_i > 0$, then $\overline{u}''(z)\,[\overline{u}(z)-\overline{u}_s] < 0$ somewhere in the field of flow, where z_s is a point at which $\overline{u}''(z)$ vanishes and $\overline{u}_s = \overline{u}(z_s)$. In particular, if $c_i > 0$ and if $\overline{u}(z)$ is monotonic and $\overline{u}''(z)$ vanishes once only in $[a,b]$, then the inequality

$$\overline{u}''(z)\,[\overline{u}(z)-\overline{u}_s] \leqslant 0$$

must hold for *all* z in $[a,b]$.

(iii) Semicircle theorem. If $c_i > 0$,

$$[c_r - \tfrac{1}{2}(\overline{u}_{\max} + \overline{u}_{\min})]^2 + c_i^2 \leqslant [\tfrac{1}{2}(\overline{u}_{\max} - \overline{u}_{\min})]^2.$$

(iv) Høiland's criterion. $\alpha c_i \leqslant \tfrac{1}{2}\max|\overline{u}'|$.

(v) If $\overline{u}'' = 0$ when $\overline{u} = \overline{u}_s$ and if $K(z) \equiv -\overline{u}''/(\overline{u}-\overline{u}_s) \geqslant 0$ throughout the flow, there is no solution with $c_i > 0$ when

$$\alpha^2 > \max\left\{\int_a^b (-|\phi'|^2 + K|\phi|^2)\,dz \Big/ \int_a^b |\phi|^2\,dz\right\} \equiv \alpha_s^2,$$

the maximum being with respect to functions ϕ which vanish at a and b and which have square-integrable derivatives. This criterion yields an upper bound for the wavenumbers at which unstable disturbances may arise. Further, by virtue of the integral inequality

$$\int_a^b |\phi'|^2\,dz \geqslant \frac{\pi^2}{(b-a)^2}\int_a^b |\phi|^2\,dz,$$

it follows that there can be no unstable disturbance with

$$\alpha^2 > \max_{\alpha}\{K\} - \pi^2/(b-a)^2; \qquad \text{(v)}'$$

clearly, the flow must be stable when the boundaries are sufficiently close for the right-hand side to be negative.

(vi) A result which has apparently not been stated previously, but which is easy to prove, is

$$c_i \leqslant \tfrac{1}{2}\max_z|\overline{u}''|\left(\alpha^2 + \frac{\pi^2}{(b-a)^2}\right)^{-1}.$$

It follows from the relationship

$$\int_a^b (|\phi'|^2 + \alpha^2|\phi|^2)\,dz = -\int_a^b \frac{\overline{u}''}{\overline{u}-c}|\phi|^2\,dz.$$

For, since the left-hand side is real, we must have

$$-\int_a^b \frac{\overline{u}''}{\overline{u}-c}|\phi|^2\,dz = -\int_a^b \frac{\overline{u}''(\overline{u}-c_r)}{(\overline{u}-c_r)^2+c_i^2}|\phi|^2\,dz \leqslant \max_z|\overline{u}''|\int_a^b \frac{|\overline{u}-c_r|\,|\phi|^2}{(\overline{u}-c_r)^2+c_i^2}\,dz$$

$$\leqslant \frac{1}{2c_i}\max_z|\overline{u}''|\int_a^b |\phi|^2\,dz,$$

and the result is obtained on invoking the above integral inequality. A result of Sattinger (1967), that

$$c_i \leqslant \left\{\frac{(\overline{u}_{\max} - \overline{u}_{\min})\max_z|\overline{u}''|}{\alpha^2 + \pi^2(b-a)^{-2}}\right\}^{\frac{1}{2}},$$

is less strong than the bound just derived, as may be seen on squaring and using the result from (iii) that $c_i \leqslant \tfrac{1}{2}(\overline{u}_{\max} - \overline{u}_{\min})$.

(vii) For flows such that $\overline{u}''/(\overline{u}-d) \leqslant 0$ everywhere for some real number d, Drazin & Howard (1962) have shown that $c_i > 0$ only if

$$\alpha \leqslant \frac{1}{2}\int_a^b |\overline{u}''/(\overline{u}-d)|\,dz, \qquad \text{(vii}a\text{)}$$

and, for unbounded flows only, that $c_i > 0$ only if

$$\alpha^3 \leqslant \frac{1}{4}\int_{-\infty}^{\infty} [\bar{u}''/(\bar{u}-d)]^2\,dz \tag{vii$b$}$$

(provided $\bar{u}$ and d are such that the respective integrals exist).

The present paper concerns the derivation of further eigenvalue bounds which, as is demonstrated by several examples, are sometimes better than those stated above. In addition, corresponding results for stratified flows are discussed.

2. An integral inequality

The Rayleigh equation and boundary conditions may be re-expressed as

$$\omega \equiv \phi'' - \alpha^2\phi = \frac{\bar{u}''}{\bar{u}-c}\phi, \quad \phi(a) = \phi(b) = 0,$$

where $\omega(z)$ is related to the perturbation vorticity. We now seek to express ϕ in terms of ω, by means of a suitable Green's function $G(z,z')$, as

$$\phi(z) = \int_a^b \omega(z')\,G(z,z')\,dz'.$$

When a and b are finite it can readily be found that

$$G(z,z') = \begin{cases} \dfrac{\sinh\alpha(z'-a)\sinh\alpha(z-b)}{\alpha\sinh\alpha(b-a)} & (a \leqslant z' < z \leqslant b), \\[2ex] \dfrac{\sinh\alpha(z-a)\sinh\alpha(z'-b)}{\alpha\sinh\alpha(b-a)} & (a \leqslant z < z' \leqslant b). \end{cases} \tag{2.1}$$

Also, if $a = -\infty$ and $b = \infty$, that is, if the flow is of unbounded extent,

$$G(z,z') = \begin{cases} (-1/2\alpha)\,e^{\alpha(z'-z)} & (-\infty \leqslant z' < z \leqslant \infty), \\ (-1/2\alpha)\,e^{\alpha(z-z')} & (-\infty \leqslant z < z' \leqslant \infty). \end{cases} \tag{2.2}$$

Further, for semi-infinite flows with $a = 0$ and $b = \infty$

$$G(z,z') = \begin{cases} (-1/\alpha)\,e^{-\alpha z}\sinh\alpha z' & (0 \leqslant z' < z \leqslant \infty), \\ (-1/\alpha)\,e^{-\alpha z'}\sinh\alpha z & (0 \leqslant z < z' \leqslant \infty). \end{cases} \tag{2.3}$$

The Rayleigh equation and boundary conditions may then be expressed as the integral equation

$$\omega(z) = \frac{\bar{u}''(z)}{\bar{u}(z)-c}\int_a^b G(z,z')\,\omega(z')\,dz'. \tag{2.4}$$

Suppose now that $\omega(z)$ belongs to the space of complex-valued functions $L_p[a,b]$ such that

$$\|f\|_p \equiv \left(\int_a^b |f(z)|^p\,dz\right)^{1/p} < \infty \quad (p \geqslant 1),$$

the integral being in the Lebesgue sense. From (2.4),

$$\|\omega\|_p \equiv \left(\int_a^b |\omega(z)|^p\,dz\right)^{1/p} = \left(\int_a^b \left|\frac{\bar{u}''(z)}{\bar{u}(z)-c}\int_a^b G(z,z')\,\omega(z')\,dz'\right|^p dz\right)^{1/p}.$$

Invoking the Hölder inequality

$$\int_a^b X(t)\,Y(t)\,dt \leqslant \left(\int_a^b |X(t)|^p\,dt\right)^{1/p}\left(\int_a^b |Y(t)|^q\,dt\right)^{1/q} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right),$$

with $\qquad t = z', \quad X = \omega, \quad Y = [\bar{u}''(z)/(\bar{u}(z)-c)]\,G(z,z'),$

and recalling that $G(z,z') \leqslant 0$ for all z and z' in $[a,b]$, we have

$$\|\omega\|_p \leqslant \|\omega\|_p \left(\int_a^b \left|\frac{\bar{u}''(z)}{\bar{u}(z)-c}\right|^p \left(\int_a^b |G(z,z')|^q\,dz'\right)^{p/q} dz\right)^{1/p}.$$

On cancelling $\|\omega\|_p$, this yields the inequality

$$\left(\int_a^b \left|\frac{\bar{u}''(z)}{\bar{u}(z)-c}\right|^p \left(\int_a^b |G(z,z')|^q\,dz'\right)^{p/q} dz\right)^{1/p} \geqslant 1 \quad \left(\frac{1}{p}+\frac{1}{q}=1, \quad 1 \leqslant p \leqslant \infty\right). \tag{2.5}$$

That is to say, given $\bar{u}(z)$, this inequality is satisfied by the complex eigenvalues $c(\alpha)$ associated with all eigenfunctions for which $\omega(z)$ is in the space $L_p[a,b]$.

We now investigate the conditions under which ω is in $L_p[a,b]$. It seems reasonable to assume that, for most eigenfunctions of interest, $\omega(z)$ will be in $L_s[a,b]$ for at least *some* value of $s \geqslant 1$. If this is so, the function

$$\int_a^b G(z,z')\,\omega(z')\,dz' \tag{2.6}$$

is bounded and continuous. If we also suppose that $\bar{u}''/(\bar{u}-c)$ is in the space $L_p[a,b]$ it follows from (2.4) that ω is also in $L_p[a,b]$, since the product of an L_p function and a bounded continuous function is also in L_p space. Accordingly, we have shown that, if ω is in $L_s[a,b]$ for *some* $s \geqslant 1$ and if $\bar{u}''/(\bar{u}-c)$ is in $L_p[a,b]$, then ω is also in $L_p[a,b]$.

This result may be strengthened to allow ω and $\bar{u}''/(\bar{u}-c)$ to contain a linear combination of delta functions, with the integrals appropriately interpreted. For, if we suppose ω to be of the form

$$f_s(z) + \sum_{n=1}^{N} A_n\,\delta(z-z_n), \tag{2.7}$$

where the integer N and the constants A_n are finite and f_s is in $L_s[a,b]$ for some $s \geqslant 1$, the expression (2.6) is again bounded and continuous. Hence, if $\bar{u}''/(\bar{u}-c)$ is of the form

$$f_p(z) + \sum_{n=1}^{N'} B_n\,\delta(z-z_n),$$

with the integer N' and the constants B_n finite and f_p in $L_p[a,b]$, then so also is ω.

If $\bar{u}''/(\bar{u}-c)$ does *not* belong to the class of functions described above the left-hand side of (2.5) is unbounded and the inequality is automatically satisfied. The inequality (2.5) therefore holds subject only to a very weak assumption concerning the permissible eigenfunctions: namely, that ω is of the form (2.7) for at least *one* $s \geqslant 1$. If ω were *not* of this form, the right-hand side of (2.4) would be meaningless in any case.

3. The adjoint equation

The equation
$$(\bar{u}-c)(\phi''-\alpha^2\phi)+2\bar{u}'\phi' = 0 \tag{3.1}$$
and boundary conditions
$$\phi(a) = \phi(b) = 0$$

define the eigenvalue problem adjoint to that discussed above. Accordingly, the eigenvalues for the two problems are identical, and we may proceed as above, but using equation (3.1) instead of (1.1), to derive another integral inequality of interest. Now, we wish to express ϕ' in terms of ω, by means of a suitable Green's function $H(z,z')$, as

$$\phi'(z) = \int_a^b \omega(z')\,H(z,z')\,dz'.$$

With a and b finite, we find

$$H(z,z') = \begin{cases} \dfrac{\sinh\alpha(z'-a)\cosh\alpha(z-b)}{\sinh\alpha(b-a)} & (a \leqslant z' < z \leqslant b), \\ \dfrac{\sinh\alpha(z'-b)\cosh\alpha(z-a)}{\sinh\alpha(b-a)} & (a \leqslant z < z' \leqslant b), \end{cases} \tag{3.2}$$

while with $a = -\infty$ and $b = \infty$

$$H(z,z') = \begin{cases} \tfrac{1}{2}e^{\alpha(z'-z)} & (-\infty \leqslant z' < z \leqslant \infty), \\ -\tfrac{1}{2}e^{\alpha(z-z')} & (-\infty \leqslant z < z' \leqslant \infty), \end{cases} \tag{3.3}$$

and, with $a = 0$ and $b = \infty$,

$$H(z,z') = \begin{cases} e^{-\alpha z}\sinh\alpha z' & (0 \leqslant z' < z \leqslant \infty), \\ -e^{-\alpha z'}\cosh\alpha z & (0 \leqslant z < z' \leqslant \infty). \end{cases} \tag{3.4}$$

By a similar argument to that used above, we obtain the inequality

$$\left[\int_a^b \left|\frac{2\bar{u}'(z)}{\bar{u}(z)-c}\right|^p \left(\int_a^b |H(z,z')|^q\,dz'\right)^{p/q} dz\right]^{1/p} \geqslant 1 \quad \left(\frac{1}{p}+\frac{1}{q} = 1, \quad 1 \leqslant p \leqslant \infty\right), \tag{3.5}$$

which is valid under a similarly weak condition concerning the form of ω.

4. The eigenvalue bounds

Unbounded flows

When $a = -\infty$ and $b = \infty$ the Green's function $G(z,z')$ is of the form (2.2) and

$$\left(\int_{-\infty}^{\infty} |G(z,z')|^q\,dz'\right)^{1/q} = \frac{1}{2\alpha}\left(\frac{2}{\alpha q}\right)^{1/q} \quad (q > 0),$$

a result *independent of the value of* z. In this case, inequality (2.5) becomes

$$\left(\int_{-\infty}^{\infty} \left|\frac{\bar{u}''}{\bar{u}-c}\right|^p dz\right)^{1/p} \geqslant 2\alpha\left(\frac{\alpha p}{2(p-1)}\right)^{(p-1)/p} \tag{4.1}$$

on setting q^{-1} equal to $1-p^{-1}$. *A fortiori*, this yields an upper bound for c_i, the imaginary part of c, namely

$$\left(\int_{-\infty}^{\infty} |\bar{u}''|^p \, dz\right)^{1/p} \geqslant 2\alpha c_i \left(\frac{\alpha p}{2(p-1)}\right)^{(p-1)/p}. \qquad (4.1a)$$

We recall that these results hold for all p in the range $1 \leqslant p \leqslant \infty$, the case $p = \infty$ yielding

$$\frac{1}{c_i} \max_z |\bar{u}''| \geqslant \max_z \left|\frac{u''}{\bar{u}-c}\right| \geqslant \alpha^2.$$

Similarly, for the adjoint equation,

$$\left(\int_{-\infty}^{\infty} |H(z,z')|^q \, dz'\right)^{1/q} = \frac{1}{2}\left(\frac{2}{\alpha q}\right)^{1/q} \quad (q > 0)$$

from (3.3), and the inequality (3.5) becomes

$$\left(\int_{-\infty}^{\infty} \left|\frac{\bar{u}'}{\bar{u}-c}\right|^p dz\right)^{1/p} \geqslant \left(\frac{\alpha p}{2(p-1)}\right)^{(p-1)/p}, \qquad (4.2)$$

or, *a fortiori*,

$$\left(\int_{-\infty}^{\infty} |\bar{u}'|^p \, dz\right)^{1/p} \geqslant c_i \left(\frac{\alpha p}{2(p-1)}\right)^{(p-1)/p}. \qquad (4.2a)$$

Thus we have derived two families of eigenvalue bounds: (4.1) and (4.2). For a particular velocity profile, a search may be made using these for that value of p which yields the strongest result. Some illustrations of this are given in § 5.

Semi-infinite flows

With $a = 0$ and $b = \infty$, the appropriate Green's function $G(z,z')$ is (2.3) and it is readily shown that

$$\left(\int_0^{\infty} |G(z,z')|^q \, dz'\right)^{1/q} = \frac{e^{-\alpha z}}{\alpha}\left(\int_0^z \sinh^q \alpha z' \, dz' + \frac{1}{\alpha q}\sinh^q \alpha z\right)^{1/q} \leqslant \frac{1}{2\alpha}\left(\frac{2}{\alpha q}\right)^{1/q}$$

for all $q > 0$ and all z in the range $0 \leqslant z \leqslant \infty$. Similarly, it is found that, with $H(z,z')$ given by (3.4),

$$\left(\int_0^{\infty} |H(z,z')|^q \, dz'\right)^{1/q} = e^{-\alpha z}\left(\int_0^z \sinh^q \alpha z' \, dz' + \frac{1}{\alpha q}\cosh^q \alpha z\right)^{1/q} \leqslant \left(\frac{1}{q\alpha}\right)^{1/q}.$$

The inequalities (2.5) and (3.5) then yield the bounds

$$\frac{1}{c_i}\left(\int_0^{\infty} |\bar{u}''|^p \, dz\right)^{1/p} \geqslant \left(\int_0^{\infty} \left|\frac{\bar{u}''}{\bar{u}-c}\right|^p dz\right)^{1/p} \geqslant 2\alpha\left(\frac{\alpha p}{2(p-1)}\right)^{(p-1)/p} \qquad (4.3)$$

and

$$\frac{1}{c_i}\left(\int_0^{\infty} |\bar{u}'|^p \, dz\right)^{1/p} \geqslant \left(\int_0^{\infty} \left|\frac{\bar{u}'}{\bar{u}-c}\right|^p dz\right)^{1/p} \geqslant \frac{1}{2}\left(\frac{\alpha p}{p-1}\right)^{(p-1)/p}, \qquad (4.4)$$

which are analogous to those above.

Bounded flows

When a and b are finite, with $b > a$, $G(z,z')$ and $H(z,z')$ are given by (2.1) and (3.2) respectively. Then,

$$\left(\int_a^b |G(z,z')|^q\,dz'\right)^{1/q}$$
$$= \frac{\left(|\sinh\alpha(z-b)|^q\int_a^z |\sinh\alpha(z'-a)|^q\,dz' + |\sinh\alpha(z-a)|^q\int_z^b |\sinh\alpha(b-z')|^q\,dz'\right)^{1/q}}{\alpha\sinh\alpha(b-a)}$$
$$\leqslant \left(2\int_0^{\frac{1}{2}(b-a)} \sinh^q\alpha\zeta\,d\zeta\right)^{1/q}\Big/ 2\alpha\cosh\left[\tfrac{1}{2}\alpha(b-a)\right]$$

and

$$\left(\int_a^b |H(z,z')|^q\,dz'\right)^{1/q}$$
$$= \frac{\left(|\cosh\alpha(z-b)|^q\int_a^z |\sinh\alpha(z'-a)|^q\,dz' + |\cosh\alpha(z-a)|^q\int_z^b |\sinh\alpha(b-z')|^q\,dz'\right)^{1/q}}{\sinh\alpha(b-a)}$$
$$\leqslant \left(2\int_0^{\frac{1}{2}(b-a)} \sinh^q\alpha\zeta\,d\zeta\right)^{1/q}\Big/ 2\sinh\left[\tfrac{1}{2}\alpha(b-a)\right].$$

The inequalities (2.5) and (3.5) therefore give

$$\frac{1}{c_i}\left(\int_a^b |\overline{u}''|^p\,dz\right)^{1/p} \geqslant \left(\int_0^\infty \left|\frac{\overline{u}''}{\overline{u}-c}\right|^p dz\right)^{1/p} \geqslant \frac{2\alpha\cosh\left[\frac{1}{2}\alpha(b-a)\right]}{\left(2\int_0^{\frac{1}{2}(b-a)}\sinh^q\alpha\zeta\,d\zeta\right)^{1/q}}, \tag{4.5}$$

$$\frac{1}{c_i}\left(\int_a^b |\overline{u}'|^p\,dz\right)^{1/p} \geqslant \left(\int_a^b \left|\frac{\overline{u}'}{\overline{u}-c}\right|^p dz\right)^{1/p} \geqslant \frac{\sinh\left[\frac{1}{2}\alpha(b-a)\right]}{\left(2\int_0^{\frac{1}{2}(b-a)}\sinh^q\alpha\zeta\,d\zeta\right)^{1/q}}, \quad \frac{1}{q} = 1-\frac{1}{p}. \tag{4.6}$$

5. Wavenumber bounds

With velocity profiles $\overline{u}(z)$ for which $\overline{u}''(z)$ vanishes when $\overline{u} = \overline{u}_s$, there exists a neutral mode with $c_i = 0$ and $c_r = \overline{u}_s$, occurring at the wavenumber α_s defined in §1(v). If, also, $K(z) \equiv -\overline{u}''/(\overline{u}-\overline{u}_s) > 0$ throughout the flow, there are no unstable solutions ($c_i > 0$) of (1.1) for which $\alpha \geqslant \alpha_s$. (The latter result is that stated in §1 (v). For proofs of both the above statements see Drazin & Howard (1966, pp. 12–14).) However, for given $K(z)$, the expression (v) for α_s is not readily evaluated, since it involves maximizing over the class of functions ϕ which vanish at the boundaries and which have square-integrable derivatives. For this reason it is of value to have upper bounds for α_s which are more amenable to calculation for velocity profiles of interest. One such bound may be deduced from result (v)′ in §1, namely, $\alpha_s^2 \leqslant \max\{K\} - \pi^2/(b-a)^2$. The inequalities (4.1)–(4.6) above give other such bounds.

To avoid possible misunderstanding it seems worth stating that, if $K(z) \geqslant 0$ throughout the flow, the bounds to be derived yield values *above which there are* *no wavenumbers α corresponding to unstable solutions.* However, if $K(z) < 0$ somewhere in the flow, the bounds apply only to the neutral mode. That is, they give values *above which there are no wavenumbers α corresponding to neutral solutions with* $c_r = \overline{u}_s$ *and* $c_i = 0$; the possibility of neutral or unstable solutions with other values of c_r then remains.

The bounds in question are obtained immediately, by setting $c = \overline{u}_s$ and $\alpha = \alpha_s$ in results (4.1)–(4.6). With $K(z) \equiv -\overline{u}''/(\overline{u}-\overline{u}_s)$ and $L(z) \equiv \overline{u}'/(\overline{u}-\overline{u}_s)$, they may be written as follows, with $1 \leqslant p \leqslant \infty$.

Unbounded flows

$$\left(\int_{-\infty}^{\infty} |K|^p\,dz\right)^{1/p} \geqslant 2\alpha_s\left(\frac{\alpha_s p}{2(p-1)}\right)^{(p-1)/p}, \quad \left(\int_{-\infty}^{\infty} |L|^p\,dz\right)^{1/p} \geqslant \left(\frac{\alpha_s p}{2(p-1)}\right)^{(p-1)/p}. \tag{5.1, 2}$$

Semi-infinite flows

$$\left(\int_{0}^{\infty} |K|^p\,dz\right)^{1/p} \geqslant 2\alpha_s\left(\frac{\alpha_s p}{2(p-1)}\right)^{(p-1)/p}, \quad \left(\int_{0}^{\infty} |L|^p\,dz\right)^{1/p} \geqslant \frac{1}{2}\left(\frac{\alpha_s p}{p-1}\right)^{(p-1)/p}. \tag{5.3, 4}$$

Bounded flows

$$\left(\int_a^b |K|^p\,dz\right)^{1/p} \geqslant \frac{2\alpha_s \cosh\left[\tfrac{1}{2}\alpha_s(b-a)\right]}{\left(2\int_0^{\frac{1}{2}(b-a)} \sinh^q \alpha_s\zeta\,d\zeta\right)^{1/q}}, \tag{5.5}$$

$$\left(\int_a^b |L|^p\,dz\right)^{1/p} \geqslant \frac{\sinh\left[\tfrac{1}{2}\alpha_s(b-a)\right]}{\left(2\int_0^{\frac{1}{2}(b-a)} \sinh^q \alpha_s\zeta\,d\zeta\right)^{1/q}}, \tag{5.6}$$

with $1/q = 1-1/p$.

6. Examples

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In order to demonstrate the values of the above bounds, as compared with those described in § 1, a number of examples are examined. For some of these, the exact solution of the stability problem is known; it is then possible to see just how good – or bad – all these bounds are.

(*a*) For the flow

$$\overline{u}(z) = \begin{cases} 1+\lambda(z/a-1) & (a < z \leqslant \infty), \\ z/a & (|z| < a), \\ -1+\lambda(z/a+1) & (-\infty \leqslant z < -a), \end{cases}$$

with $\overline{u}''(z)$ defined as $\lambda[\delta(z-a)-\delta(z+a)]$, where $\delta(z)$ denotes the Dirac delta function and λ and a are constants, only one of the criteria stated in § 1 yields a non-trivial result. This is Høiland's criterion (iv), which gives

$$\alpha c_i \leqslant \begin{cases} 1/2a & (\lambda \leqslant 1), \\ \lambda/2a & (\lambda > 1), \end{cases}$$

(provided $\overline{u}'$ is suitably defined at $z = \pm a$). Of the bounds derived above for c_i, one also yields a non-trivial result. This is (4.1 *a*) with $p = 1$, which gives

$$\alpha c_i \leqslant |1-\lambda|/a,$$

a result better than Høiland's when $\frac{1}{2} < \lambda < 2$.

(*b*) When

$$|\bar{u}''| = (1+z^2)^{-\frac{1}{2}} \quad (0 < |z| \leqslant \infty),$$

a corresponding family of velocity profiles is

$$\bar{u} = \begin{cases} 1+z\sinh^{-1}z-(z^2+1)^{\frac{1}{2}}+bz & (0 \leqslant z \leqslant \infty), \\ \pm[1+z\sinh^{-1}z-(z^2+1)^{\frac{1}{2}}]+bz & (-\infty \leqslant z < 0), \end{cases}$$

where $\bar{u}'(0) \equiv b$ is constant, $\bar{u}'' > 0$ for $z > 0$, and the $\pm$ sign is taken so as to agree with the sign of $\bar{u}''$ for $z < 0$. Criterion (ii) of § 1 shows that, if $\bar{u}'' < 0$ for $z < 0$ and $\bar{u}''$ is *defined* as zero at $z = 0$, then no unstable disturbance is possible when $b \geqslant 0$.

For $b < 0$, the only criterion of § 1 which applies is (vi), giving $c_i \leqslant (2\alpha^2)^{-1}$. On the other hand, result (4.1 *a*) yields finite bounds for all p greater than unity. The case $p = \infty$ gives $c_i \leqslant \alpha^{-2}$, which is less good than (vi), but for all other cases with $p > 1$ our result is better than (vi) for sufficiently small α, since the power of α in (4.1 *a*) exceeds -2 for all finite $p > 1$. These bounds are

$$c_i \leqslant \tfrac{1}{2}\{B[\tfrac{1}{2}(p-1), \tfrac{1}{2}]\}^{1/p}\left(\frac{2(p-1)}{p}\right)^{(p-1)/p}\alpha^{(1-2p)/p},$$

where $B[u, v]$ denotes the Beta function. For example, with $p = 2$, we have $c_i \leqslant \pi^{\frac{1}{2}}/2\alpha^{\frac{3}{2}}$, which is better than (vi) whenever $\alpha < \pi^{-1}$. For given α, the best bound may be found by plotting the right-hand side as a function of p to locate the minimum. With sufficiently small values of α the optimum value of p will be close to (but greater than) unity, while, if α is very large, criterion (vi) will give the best available bound.

(*c*) The anti-symmetric profile

$$\bar{u} = \operatorname{sgn}(z)\int_0^{|z|}\frac{\zeta^{\frac{1}{2}}\,d\zeta}{1+\zeta} \quad (-\infty \leqslant z \leqslant \infty)$$

has

$$\bar{u}' = |z|^{\frac{1}{2}}/(1+|z|).$$

Criteria (iii) and (v)′ here yield only trivial results, but (iv) gives $c_i \leqslant \frac{1}{4}\alpha^{-1}$. Result (4.1 *a*) yields non-trivial results for $1 \leqslant p < 2$, when the bounds for c_i are proportional to powers of α in the range α^{-1} to $\alpha^{-\frac{3}{2}}$. These bounds will therefore improve upon (iv) for α sufficiently large. On the other hand, result (4.2 *a*) with $2 < p < \infty$ yields bounds for c_i which will improve upon (iv) for α sufficiently *small*, the range of powers of α being $\alpha^{-\frac{1}{2}}$ to α^{-1}. These latter bounds are

$$c_i \leqslant \left[\frac{2(p-1)}{\alpha p}\right]^{(p-1)/p}\left[\frac{2p(\Gamma(\frac{1}{2}p))^2}{(p-2)\,\Gamma(p)}\right]^{1/p} \quad (p > 2).$$

For instance, with $p = 3$, the bound is $(\frac{4}{3}\pi)^{\frac{1}{3}}\alpha^{-\frac{2}{3}}$, which is better than (iv) for $\alpha < 3/(256\pi)$.

(*d*) For the Bickley jet

$$\bar{u} = \operatorname{sech}^2 z \quad (-\infty \leqslant z \leqslant \infty),$$

there are two neutral modes with $c_r = \frac{2}{3}$ – one sinuous, the other varicose – occurring at $\alpha = 2$ and 1 respectively. Corresponding values of c_r, c_i and α for the sinuous and varicose modes are tabulated by Drazin & Howard (1966, p. 41). In this case, the criteria (iii), (iv) and (vi) of § 1 give

$$c_i \leqslant \tfrac{1}{2}, \quad c_i \leqslant (2/3\sqrt{3})\,\alpha^{-1}, \quad c_i \leqslant \alpha^{-2}$$

respectively. It turns out that, although the bounds for c_i obtained from (4.1*a*) and (4.2*a*) may be better than one or two of these results for particular values of α and p, they are never better than all three: that is to say, the bounds (4.1*a*) and (4.2*a*) for c_i do not improve upon the criteria of § 1 for this velocity profile.

Nevertheless, the wavenumber bounds of § 5 do yield new results. Criterion (v)′ of § 1 shows that there are no unstable disturbances with $\alpha^2 > 6$ and this result is reproduced by (5.1) with $p = \infty$. Also, from result (vii*a*) of § 1, Drazin & Howard (1962) showed that there are no unstable disturbances with $\alpha \geqslant 2{\cdot}29$ for this profile; (5.1) reproduces this result when $p = 2$. Result (5.2) yields nothing of interest, but improved bounds are obtained from (5.1) with p finite. Letting α_1 denote the wavenumber above which no unstable disturbance can occur, we have

$$\alpha_1 = \left(\frac{\pi^{\frac{1}{2}}\Gamma(p)}{\Gamma(p+\frac{1}{2})}\right)^{1/(2p-1)} 3^{p/(2p-1)} \left(\frac{2(p-1)}{p}\right)^{(p-1)/(2p-1)}, \tag{6.1}$$

the sharpest result occurring at about $p = 3{\cdot}5$, when $\alpha_1 = 2{\cdot}191$. This is quite close to the exact solution $\alpha = 2{\cdot}0$ for the sinuous neutral mode.

Since the eigenfunction $\phi(z)$ for the varicose mode must vanish at $z = 0$, upper bounds for the wavenumber of this neutral mode are given by (5.3), since the problem is identical to that for a fluid bounded at $z = 0$. Denoting these bounds by α_2, it is readily seen that

$$\alpha_2 = \alpha_1(\tfrac{1}{2})^{1/(2p-1)},$$

with α_1 given above. The best results are for p between $1{\cdot}5$ and 2, when $\alpha_2 \simeq 1{\cdot}82$. Since the exact solution is $\alpha = 1{\cdot}0$ this bound is much less sharp than that for the sinuous mode.

(*e*) The inviscid stability of the shear layer

$$\bar{u} = \tanh z \quad (-\infty \leqslant z \leqslant \infty)$$

has been determined by Michalke (1964), the neutral mode at the stability boundary having $c_r = 0$ and $\alpha = 1$. Again, the criteria (iii), (iv) and (vi) of § 1, taken together, are better than the bounds for c_i given by (4.1*a*) and (4.2*a*).

On setting $\bar{u}_s = 0$, criterion (v)′ shows that there is no unstable disturbance with α greater than $\sqrt{2}$, as does result (5.1) when $p = \infty$. For finite p, (5.1) improves on this bound, the wavenumber α_3 above which no unstable disturbance can exist being given by

$$\alpha_3 = (\tfrac{1}{3})^{p/(2p-1)}\,\alpha_1,$$

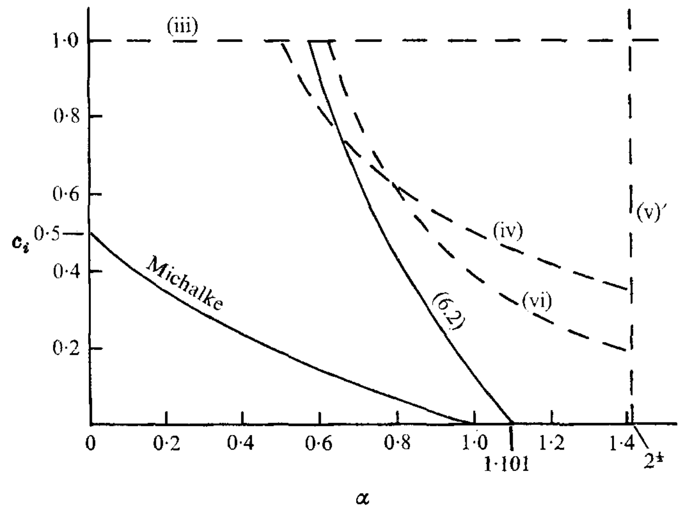

FIGURE 1. Eigenvalue bounds for $\overline{u} = \tanh z$ and Michalke's exact solution. Curves labelled (iii)–(vi) denote the criteria thus designated in § 1. That labelled (6.2) corresponds to equation (6.2).

where α_1 is defined in (6.1). The smallest value of α_3 occurs close to $p = 1{\cdot}9$, when $\alpha_3 = 1{\cdot}10$. This compares well with the exact solution $\alpha = 1{\cdot}0$ (as also does (vii*b*) of § 1 since it corresponds to $p = 2$).

For the present anti-symmetric profile, still better bounds for c_i can be derived directly from (4.1), since it is known that c_r remains zero when $c_i \neq 0$ (see Tatsumi & Gotoh 1960). On setting c equal to ic_i in (4.1), the integral may be evaluated directly for the case $p = 2$ (which is close to the best result when $c_i = 0$) to give the inequality

$$\alpha < \alpha_4 = 2^{\frac{1}{2}}[c_i^2 + \tfrac{2}{3} - c_i(1 + c_i^2)\cot^{-1} c_i]^{\frac{1}{2}}. \tag{6.2}$$

The curve of α_4 against c_i is shown in figure 1, together with Michalke's exact solution and the bounds given by (iii), (iv), (v)′ and (vi) of § 1. It is seen that a considerable improvement is obtained over the bounds of § 1.

7. Eigenvalue bounds for stratified flow

For incompressible inviscid flow of variable density $\overline{\rho}(z)$, instability is described by the eigenvalue problem

$$(\overline{u} - c)(\phi'' - \alpha^2\phi) - \overline{u}''\phi + J(z)\,\phi/(\overline{u} - c) = 0,$$

$$\phi(a) = \phi(b) = 0,$$

provided the Boussinesq approximation is justified. Here $J(z) \equiv -l^2 g\overline{\rho}'/\overline{\rho}V^2$, where l and V are characteristic length and velocity scales respectively, g is gravitational acceleration and z is measured vertically upwards. For this situation, the semicircle theorem (iii) remains valid provided $J \geqslant 0$ everywhere.

Further bounds on c_i, due to Synge and Howard respectively, are

$$c_i \leqslant \max_z (2J/\bar{u}''), \quad \alpha^2 c_i^2 \leqslant \max_z (\tfrac{1}{4}\bar{u}'^2 - J) \qquad (7.1, 2)$$

(see Drazin & Howard (1966, § 5) for details).

Additional bounds may be derived by following the procedure of §§ 3–5 above, but with $\bar{u}''$ replaced by $\bar{u}'' - J/(\bar{u}-c)$ throughout. For example, with unbounded flows, the result analogous to (4.1) is

$$\left(\int_{-\infty}^{\infty} \left|\frac{\bar{u}'' - J/(\bar{u}-c)}{(\bar{u}-c)}\right|^p dz\right)^{1/p} \geqslant 2\alpha\left(\frac{\alpha p}{2(p-1)}\right)^{(p-1)/p} \quad (1 \leqslant p \leqslant \infty), \qquad (7.3)$$

and, since by Minkowski's inequality the left-hand side is less than

$$c_i^{-1}\|\bar{u}''\|_p + c_i^{-2}\|J\|_p \quad \text{for} \quad c_i > 0,$$

where $$\|\bar{u}''\|_p \equiv \left(\int_{-\infty}^{\infty} |\bar{u}''|^p\, dz\right)^{1/p}, \quad \|J\|_p = \left(\int_{-\infty}^{\infty} |J|^p\, dz\right)^{1/p},$$

we have the inequality

$$c_i < 2\lambda^{-1}[(\|\bar{u}''\|_p^2 + \lambda\|J\|_p)^{\frac{1}{2}} + \|\bar{u}''\|_p], \quad \lambda \equiv 8\alpha\left(\frac{\alpha p}{2(p-1)}\right)^{(p-1)/p}. \qquad (7.4)$$

However, this family of bounds for c_i appears to be rather weak, since they take no account of whether the stratification is stable or unstable.

Results of somewhat more interest may be derived for the neutral mode with $c_r = \bar{u}_s$, $c_i = 0$ and $\alpha = \alpha_s$. Upper bounds for α_s corresponding to (5.1) are

$$\left(\int_{-\infty}^{\infty} \left|\frac{\bar{u}''(\bar{u}-\bar{u}_s) - J}{(\bar{u}-\bar{u}_s)^2}\right|^p dz\right)^{1/p} \geqslant 2\alpha_s\left(\frac{\alpha_s p}{2(p-1)}\right)^{(p-1)/p} \quad (1 \leqslant p \leqslant \infty). \qquad (7.5)$$

However, the present bounds apply only to the neutral mode (see Howard 1963), and no conclusions can be drawn concerning the existence of unstable disturbances.

For many cases of interest, the left-hand side of (7.5) is unbounded for all values of p – this happens, for example, whenever $|u-u_s|$ and J remain finite as $z \to \pm\infty$ – but this difficulty is not encountered in the corresponding results for bounded flow, which are obtained from (4.5) on replacing $\bar{u}''$ by $\bar{u}'' - J/(\bar{u}-c)$. Examples for which (7.5) yields non-trivial bounds are examined in the next section.

8. Examples for stratified flow

We shall demonstrate only result (7.5) for two cases.

(*f*) The case

$$\bar{u} = \operatorname{sech} z \tanh z, \quad J = J_0 \operatorname{sech}^2 z \tanh^2 z \quad (-\infty \leqslant z \leqslant \infty)$$

has apparently not been studied previously. Nevertheless, it has simple solutions, corresponding to neutral varicose and sinuous modes. These are

$$\phi_1 = \operatorname{sech}^2 z, \qquad \alpha^2 = J_0 + 3, \quad c = 0,$$

$$\phi_2 = \operatorname{sech} z \tanh z, \quad \alpha^2 = J_0, \qquad c = 0.$$

For the special case $J_0 = 1$, (7.5) yields non-trivial results with $\bar{u}_s = 0$. It turns out that the bounds are then identical with α_1 defined in (6.1), the best result being close to $p = 3{\cdot}5$, when $\alpha_1 = 2{\cdot}191$. The exact result for the sinuous mode is $2{\cdot}0$; so agreement is again quite good.

(*g*) An example studied by Garcia (see Drazin & Howard 1966, § 5) is

$$\bar{u} = \tanh z, \quad J = 3J_0 \operatorname{sech}^2 z \tanh^2 z \quad (-\infty \leqslant z \leqslant \infty),$$

with J_0 constant. Result (7.5) now gives

$$\alpha_s \leqslant (\tfrac{1}{3} + \tfrac{1}{2}J_0)^{p/(2p-1)}\alpha_1$$

for the neutral mode with $c_r = 0$, with α_1 given by (6.1). When $J_0 = 0$, the best result is $\alpha_s \leqslant 1{\cdot}10$ as in example (*e*), with $p = 1{\cdot}9$. For $p = 1, 2$ and ∞ respectively, we have the bounds

$$J_0 \geqslant \tfrac{1}{3}(\alpha_s - 2), \quad J_0 \geqslant \tfrac{2}{3}[(\tfrac{3}{4}\alpha_s^3)^{\frac{1}{2}} - 1], \quad J_0 \geqslant \tfrac{1}{3}(\alpha_s^2 - 2),$$

which may be compared with the exact solution for the sinuous mode:

$$J = \tfrac{1}{3}(\alpha_s^2 + \alpha_s - 2).$$

The results for $p = 2$ and $p = \infty$, when taken together, yield bounds not too distant from the exact result over the whole range of α_s and J_0.

The result analogous to (7.5) for semi-infinite flows is identical in form to (7.5), but with the lower limit of integration replaced by 0. This result may be used to obtain bounds for the varicose mode with $c = 0$ and $\alpha = \alpha_v$, since $\phi = 0$ at $z = 0$ by symmetry for this mode. These bounds are

$$\alpha_v \leqslant (\tfrac{1}{2})^{1/(2p-1)}(\tfrac{1}{3} + \tfrac{1}{2}J_0)^{p/(2p-1)}\alpha_1$$

with α_1 as given by (6.1). When $J_0 = 0$, the best result is near $p = 1{\cdot}3$, where the bound is $\alpha_v \leqslant 0{\cdot}784$; but, for $J_0 = 0$ the exact solution is $\alpha_v = 0$. For $p = 1, 2$ and ∞ respectively, the bounds for this mode are

$$J_0 \geqslant \tfrac{2}{3}(\alpha_v - 1), \quad J_0 \geqslant \tfrac{2}{3}[(\tfrac{3}{2}\alpha_v^3)^{\frac{1}{2}} - 1], \quad J_0 \geqslant \tfrac{1}{3}(\alpha_v^2 - 2),$$

and the exact solution is $$J_0 = \tfrac{1}{3}\alpha_v(\alpha_v + 3).$$

The bounds are somewhat less good for this case than for the sinuous mode.

Finally, it is perhaps worth noting that the captions in Drazin & Howard's figure 11 (*c*) (1966, p. 78) concerning the regions of stability and instability for this example are wrong. For, the sinuous mode is unstable when $J_0 > \frac{1}{3}(\alpha^2 + \alpha - 2)$ and the varicose mode is unstable when $J_0 > \frac{1}{3}\alpha(\alpha + 3)$.

9. Conclusions

New eigenvalue bounds have been derived for the stability of inviscid homogeneous and stratified flows, separate results applying to flows of infinite, semi-infinite and finite extent. For homogeneous flows examples are given which show that the upper bounds (4.1*a*) and (4.2*a*) for the imaginary part c_i of the complex phase velocity sometimes improve upon the bounds known previously. This is particularly likely to be so in cases where one or more of the criteria (iii), (iv) and

(vi) of § 1 yields trivial results. For unbounded stratified flows the upper bounds (7.4) for c_i appear to be rather weak compared with previous known results.

The upper bounds (5.1) and (5.2) for the wavenumbers of neutrally stable disturbances may represent a considerable improvement on results (v)′ and (vii) of § 1 for homogeneous flows, and the corresponding bounds (7.5) for stratified flows appear to have no counterpart in previous work. In the examples studied, bounds within 10 % of the known exact solutions were obtained. For antisymmetric velocity profiles, such as example (*e*), further improvements may be made in the bounds for c_i by direct integration of the expressions (4.1) and (4.2) for disturbances with $c_r = 0$.

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